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ELECTROELASTIC EQUILIBRIUM OF A PIEZOCERAMIC PLATE

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Mechanical and electric fields in a piezoceramic plate are studied with the help of asymptotic integration of the three-dimensional equations of electroelasticity. It is established that the electroelastic state of the plate can be separated into the internal state, and a boundary layer-type state. Derivation of the solution of the boundary layer-type state is reduced to an infinite system. A boundary value problem is formulated in the first approximation in order to establish the internal electroelastic state of the plate.

1. Let $\Omega = S \times [-h, h]$ denote the region occupied by the plate, where S is the middle surface and $2h$ its thickness, ∂S is the boundary of S, $\Gamma = \partial S \times [-h, h]$ is its lateral surface, $S_\pm$ are the plate ends and a is the characteristic dimension of S. The plate is referred to the Cartesian $ox_1x_2x_3$ - coordinate system with the origin on S and the ox_3-axis orthogonal to S.

We assume that the material has been previously polarized along the plate thickness and, that its electroelastic properties are described by the relations [1]

$$\begin{aligned}
\sigma_{11} &= c_{11}^E s_{11} + c_{12}^E s_{22} + c_{13}^E s_{33} - e_{31}E_3 \\
\sigma_{22} &= c_{12}^E s_{11} + c_{11}^E s_{22} + c_{13}^E s_{33} - e_{31}E_3 \\
\sigma_{33} &= c_{13}^E (s_{11} + s_{22}) + c_{33}^E s_{33} - e_{33}E_3 \\
\sigma_{12} &= (c_{11}^E - c_{12}^E)\, s_{12} = 2c_{66}^E s_{12} \\
\sigma_{i3} &= 2c_{44}^E s_{i3} - e_{15}E_i \\
D_i &= 2e_{15}s_{i3} + \varepsilon_{11}^s E_i \quad (i = 1, 2) \\
D_3 &= e_{31}(s_{11} + s_{22}) + e_{33}s_{33} + \varepsilon_{33}^s E_3
\end{aligned} \tag{1.1}$$

Here c_{ij}^E are the moduli of elasticity, e_{ij} are the piezomoduli, ε_{ij}^s are the dielectric permeabilities, E_k denote the components of the electric field intensity vector, D_k are the components of the electric induction vector, σ_{ml} are the components of the stress tensor and s_{ml} are the components of the deformation tensor.

Supplementing the relations (1.1) with the Cauchy equations of equilibrium and the Maxwell equations

$$\sigma_{ml,l} = 0, \ D_{k,k} = 0, \ \text{rot}\,\mathbf{E} = 0 \ (\mathbf{E} = -\text{grad}\,\psi) \tag{1.2}$$

we obtain a closed system of equations in terms of the displacements u_i and the electric potential ψ, describing the electroelastic equilibrium of the plate.

Let us introduce the notation

$$u_4 = -\psi / d, \quad a_{ij} = c_{ij}^E / c, \quad b_{ij} = e_{ij}d / c, \quad \lambda_{ij} = \varepsilon_{ij}^s d^2 / c$$
$$\xi_k = x_k / a, \ \partial_k = \partial / \partial\xi_k \ (k = 1, 2), \ \Delta_0 = \partial_1^2 + \partial_2^2,$$
$$\xi = x_3 / h, \ \varepsilon = h / a$$

Here c and denote certain characteristic parameters of the plate material, with the dimensions of c_{ij}^E and E respectively; when dealing with concrete calculations, they can be chosen e.g. as follows: $c = c_{33}^E$, $|d| = |\mathbf{P}|$ where $\mathbf{P}$ is the preliminary polarization vector of the ceramic. We shall assume that the plate is surrounded by vacuum.

Let the following conditions hold at the plate ends:

$$\sigma_{i3}|_{S_\pm} = 0,\ i = 1, 2, 3;\ u_4|_{S_+} = \varepsilon^2 a\varphi = \text{const},\ u_4|_{S_-} = 0 \tag{1.3}$$

Let also the stresses and the electric charge surface density λ (n, s are the local coordinates of the contour ∂S [2]) be given on the lateral surface

$$\sigma_n|_\Gamma = cN(s, \xi),\quad \sigma_{ns}|_\Gamma = cT(s, \xi)$$

$$\sigma_{n\xi}|_\Gamma = cZ(s, \xi),\quad -D_n|_\Gamma = \frac{c}{d}\lambda(s, \xi)$$

We assume that the constant φ in the boundary conditions is not known, and this corresponds to the case of the plate ends which are fully electroded, but not closed [3]. The electrodes are assumed to be infinitely thin, therefore their influence on the elastic properties of the plate can be neglected.

2. To solve the proposed problem we use the system of solutions of the equations of electroelasticity (1.1), (1.2) which satisfy the following homogeneous conditions at the plate ends:

$$\sigma_{i3}|_{S_\pm} = 0\ (i = 1, 2, 3),\quad u_4|_{S_\pm} = 0$$

The author of [4] used the methods of [5] to construct a complete system of homogeneous solutions for a plate made of an electroelastic material, with the properties varying across the thickness. Using the results of these papers, we give the following system of homogeneous solutions for the problem under consideration:

The biharmonic solution

$$u_i^{(1)} = a\varepsilon\{\varphi_i - \partial_i[\Phi_1 + P_1\Phi_2 + \varepsilon^2\Delta_0(q_0P_2\Phi_1 + q_2F\Phi_2)]\},\ i = 1,2 \tag{2.1}$$
$$u_3^{(1)} = a\{\Phi_2 + \varepsilon^2\Delta_0[q_1P_1\Phi_1 - q_3(P_2 - P_0)\Phi_2]\}$$
$$u_4^{(1)} = a\varepsilon^2 q_4(P_2 - P_0)\Delta_0\Phi_2$$

Here $P_j(\xi)$ denote the Legendre polynomials, Φ_1 and Φ_2 are two-dimensional biharmonic functions and φ_1, φ_2 are conjugate harmonic functions connected with Φ_1 by the equation

$$\partial_1\varphi_1 = \partial_2\varphi_2 = \varkappa\Delta_0\Phi_1,\qquad F(\xi) = \xi^3 - 3\xi$$
$$q_0 = -a_{13}(2\varkappa - 1)/(3a_{33}),\quad q_1 = 3q_c$$
$$q_2 = -[(a_{44} + a_{13})g_1 + (b_{15} + b_{31})g_2 + a_{11}]/(6a_{44}),\ q_3 = q_1/3$$
$$q_4 = g_2/3,\quad \varkappa = (a_{11} - a_{13}^2/a_{33})/(a_{11} + a_{12} - 2a_{13}^2/a_{33})$$
$$g_1 = -(a_{13}\lambda_{33} + b_{31}b_{33})/(b_{33}^2 + a_{33}\lambda_{33}),\quad g_2 = (a_{33}b_{31} - a_{13}b_{33})/(b_{33}^2 + a_{33}\lambda_{33})$$

The potential solution

$$u_i^{(2)} = a\varepsilon^3 \sum_{k=1}^{\infty} a_k(\xi)\,\partial_i A_k,\ i=1,2;\quad u_3^{(2)} = -a\varepsilon^2 \sum_{k=1}^{\infty} \omega_k(\xi) A_k$$
$$u_4^{(2)} = a\varepsilon^2 \sum_{k=1}^{\infty} \gamma_k^2 \theta_k(\xi) A_k,\quad \operatorname{Re}\gamma_k > 0 \qquad (2.2)$$

$$a_k(\xi) = q_5 f_k'' - \gamma_k^2 q_6 f_k + q_7\theta_k'$$
$$\omega_k(\xi) = q_5 f_k''' - (q_6 - 2q_8)\gamma_k^2 f_k' + q_7\theta_k'' - q_9\gamma_k^2\theta_k$$

Here γ_k are the eigenvalues and $\{f_k,\ \theta_k\}$ denote the eigen pairs of functions (the analog of the Papkovich functions of the classical theory of elasticity) of the spectral problem

$$q_5 f^{\mathrm{IV}} + 2(q_8 - q_6)\gamma^2 f'' + q_{10}\gamma^4 f + q_7\theta''' + (q_{11} - q_9)\gamma^2\theta' = 0 \qquad (2.3)$$
$$q_7 f''' + (q_{11} - q_9)\gamma^2 f' + q_{12}\theta'' + q_{13}\gamma^2\theta = 0$$
$$f(\pm 1) = 0 = f'(\pm 1),\quad \theta(\pm 1) = 0$$
$$q_5 = a_{33}/g,\quad q_6 = a_{13}/g,\quad q_7 = (a_{33}b_{31} - a_{13}b_{33})/g,\ q_8 = 1/(2a_{44})$$
$$q_9 = b_{15}/a_{44},\quad q_{10} = a_{11}/g,\quad q_{11} = (a_{11}b_{33} - a_{13}b_{31})/g$$
$$q_{12} = \lambda_{33} + b_{33}q_{11} + b_{31}q_7,\quad q_{13} = \lambda_{11} + b_{15}q_9,\ g = a_{11}a_{33} - a_{13}$$

and the functions A_k in (2.2) satisfy the relation

$$(\varepsilon^2\Delta_0 - \gamma_k^2) A_k(\xi_1, \xi_2) = 0$$

The rotational solution

$$u_1^{(3)} = a\varepsilon^3 \sum_{p=1}^{\infty} t_p(\xi)\,\partial_2 B_p,\quad u_2^{(3)} = -a\varepsilon^3 \sum_{p=1}^{\infty} t_p(\xi)\,\partial_1 B_p \qquad (2.4)$$
$$u_3^{(3)} = 0 = u_4^{(3)},\quad \delta_p > 0$$

where

$$a_{44} t_p'' + a_{66}\delta_p^2 t_p = 0,\quad t_p'(\pm 1) = 0$$
$$(\varepsilon^2\Delta_0 - \delta_p^2) B_p(\xi_1, \xi_2) = 0$$

It should be noted that contrary to the elastic case [6], the spectrum $\{\gamma_k\}$ of the problem (2.3) depends on the electroelastic properties of the material. Nevertheless, for the majority of the types of piezoceramics used (PZT-4, PZT-5, $TsTC$-19, etc.) the spectrum distribution has certain common features: the spectrum $\{\gamma_k\}$ is discrete, symmetrically distributed in the complex plane, and has a point of accumulation at infinity; none of γ_k are pure imaginary; when $|\gamma| \to \infty$ $(\operatorname{Re}\gamma > 0)$ three asymptotes of the distribution of γ_k exist, one of them represented by the real axis and the other two by the straight lines $\arg\gamma = \pm\nu,\ \nu \neq 0$.

Let us give the formulas for the asymptotic values of the real and complex γ_k

$$\gamma_n = [(n-1)\pi + r\pi/2 - \alpha]/\mu_1 \qquad (2.5)$$
$$\gamma_m = -i\bar{\mu}_2 \{\ln|G_1 + iG_2| + \arg[(-1)^r (G_1 + iG_2)] + 2(m-1)\pi i\}/(2|\mu_2|)$$
$$r = 0,\ 1;\ n = 1, 2, 3, \ldots;\quad m = 1, 2, 3, \ldots$$

$$\operatorname{tg}\alpha = G_1 / G_2, \quad G_1 = X_1 \operatorname{Im}(Y_2\bar{Z}_2) / G$$
$$G_2 = \operatorname{Re}[X_2 (Y_1\bar{Z}_2 - Z_1\bar{Y}_2)] / G, \quad G = \operatorname{Im}[X_2 (Y_1\bar{Z}_2 - Z_1\bar{Y}_2)] - X_1 \operatorname{Im}(Y_2\bar{Z}_2)$$
$$\operatorname{Im}\mu_j > 0, \quad \operatorname{Re}\gamma_k > 0, \quad \operatorname{Im}\gamma_k > 0$$

The relations connecting the constants μ_j, X_j, Y_j and Z_j with the electroelastic characteristics of the plate material are given in [7].

It is significant that even the first values of γ_k $(n, m = 1)$ obtained from (2.5) differ from the exact values [8] by not more than 6%.

Let us explain briefly some properties of the homogeneous solutions. The potential and rotational solutions contain, as implied by (2.2) and (2.4), the functions A_k and B_p which represent the solutions of equations in which the parameter ε^2 accompanies the higher order derivatives. Using the properties of the spectra $\{\gamma_k\}$ and $\{\delta_p\}$ we can show that for small ε the solutions of these equations resemble a boundary layer localised at the boundary ∂S [2]. For this reason the potential and rotational solutions decay rapidly with increasing distance from the lateral surface Γ. Thus the inner electroelastic state of the plate is determined by the biharmonic solution.

The inhomogeneity apparent in the condition (1.3) can be removed using a particular solution of the form

$$u_1^{(4)} = 0 = u_2^{(4)}, \quad u_3^{(4)} = \varepsilon^2 a b_{33} \varphi \xi / (2a_{33}), \quad u_4^{(4)} = \varepsilon^2 a \varphi (\xi + 1) / 2$$

Then the general solution $u_l = u_l^{(1)} + \ldots + u_l^{(4)}$ $(l = 1, 2, 3, 4)$ will satisfy the equations of electroelasticity (1.1) and (1.2) and the boundary conditions (1.3).

3. In order to satisfy the boundary conditions at the lateral surface Γ of the plate, we use the variational principle formulated in [4]. In the present case the principle can be written in the form

$$\iint_\Gamma \Big\{(\sigma_n - N)\Big[a\Big(\delta u_{0n}^{(1)} - \varepsilon P_1 \delta \frac{\partial}{\partial n}\Phi_2\Big) + \delta u_n^{(2)} + \delta u_n^{(3)}\Big] + (\sigma_{ns} - T)\Big[a\Big(\delta u_{0s}^{(1)} + \varepsilon P_1 \frac{\partial}{\partial s}\delta\Phi_2\Big) + \delta u_s^{(2)} + \delta u_s^{(3)}\Big] + (\sigma_{n\xi} - Z)(a\delta\Phi_2 + \delta u_3^{(2)}) - (D_n + \lambda)\delta u_4^{(2)}\Big\} ds\, d\xi - \delta u_4^{(4)} \iint_{S_+} D_\xi \, d\Sigma = 0 \tag{3.1}$$

where (R is the radius of curvature of ∂S)

$$\sigma_n = \varepsilon \Big\{{}^1/{}_2\varkappa_1\varphi + 2a_{66} Q_n [\Phi_1 + P_1\Phi_2 + \varepsilon^2\Delta_0 (q_0 P_2 \Phi_1 + q_2 F \Phi_2)] - \varkappa_2 \Delta_0 \Phi_2 + \sum_{k=1}^{\infty} (\gamma_k^2 f_k'' A_k - 2\varepsilon^2 a_{66} a_k Q_n A_k) - 2\varepsilon^2 a_{66} \sum_{p=1}^{\infty} t_p Q_s B_p\Big\}$$

$$\sigma_{ns} = \varepsilon \Big\{2a_{66}Q_s[\Phi_1 + P_1\Phi_2 + \varepsilon^2\Delta_0(q_0P_2\Phi_1 + q_2F\Phi_2)] -$$

$$2\varepsilon^2 a_{66}\sum_{k=1}^{\infty} a_k Q_s A_k + \sum_{p=1}^{\infty}(a_{44}t_p'' B_p + 2\varepsilon^2 a_{66} t_p Q_n B_p)\Big\}$$

$$\sigma_{n\xi} = \varepsilon^2\Big[-\varkappa_3\frac{\partial}{\partial n}\Delta_0\Phi_2 - \sum_{k=1}^{\infty}\gamma_k^2 f_k'\frac{\partial}{\partial n}A_k + a_{44}\sum_{p=1}^{\infty}t_p'\frac{1}{H}\frac{\partial}{\partial s}B_p\Big]$$

$$D_n = \varepsilon^2\Big[-\varkappa_4\frac{\partial}{\partial n}\Delta_0\Phi_2 - \sum_{k=1}^{\infty}d_k(\xi)\frac{\partial}{\partial n}A_k + b_{15}\sum_{p=1}^{\infty}t_p'\frac{1}{H}\frac{\partial}{\partial s}B_p\Big]$$

$$D_\xi = \varepsilon\Big[\frac{1}{2}\varkappa_5\varphi + \varkappa_6\Delta_0\Phi_1 + \sum_{k=1}^{\infty}\gamma_k^2(q_7f_k'' + \gamma_k^2 q_{11}f_k + q_{12}\theta_k')A_k\Big]$$

$$u_n^{(2)} + u_n^{(3)} = a\varepsilon^3\Big(\sum_{k=1}^{\infty}a_k\frac{\partial}{\partial n}A_k + \sum_{p=1}^{\infty}t_p\frac{1}{H}\frac{\partial}{\partial s}B_p\Big)$$

$$u_s^{(2)} + u_s^{(3)} = a\varepsilon^3\Big(\sum_{k=1}^{\infty}a_k\frac{1}{H}\frac{\partial}{\partial s}A_k - \sum_{p=1}^{\infty}t_p\frac{\partial}{\partial n}B_p\Big)$$

$$Q_n(\cdot) = \Big(\frac{1}{H^2}\frac{\partial^2}{\partial s^2} + \frac{a}{RH}\frac{\partial}{\partial n} + \frac{naR'}{H^3R^2}\frac{\partial}{\partial s}\Big)(\cdot)$$

$$Q_s(\cdot) = -\Big(\frac{1}{H}\frac{\partial^2}{\partial n\,\partial s} - \frac{a}{H^2R}\frac{\partial}{\partial s}\Big)(\cdot), \quad H = 1 + naR^{-1}$$

$$\varkappa_1 = (a_{13}b_{33} - a_{33}b_{31})\,/\,a_{33}, \quad \varkappa_2 = a_{11}P_1 + (a_{13}q_3 + b_{31}q_4)P_2'$$

$$\varkappa_3 = 3\,[a_{44}q_2 + (a_{44}q_3 + b_{15}q_4)\,/\,2]\,(\xi^2 - 1), \quad \varkappa_4 = 3\,[b_{15}q_2 + (b_{15}q_3 - \lambda_{11}q_4)\,/\,2]\,(\xi^2 - 1)$$

$$\varkappa_5 = (b_{33}^2 + a_{33}\lambda_{33})\,/\,a_{33}, \quad \varkappa_6 = b_{31}(2\varkappa - 1) + b_{33}q_1$$

$$d_k(\xi) = \gamma_k^2(q_9f_k' - q_{13}\theta_k)$$

Choosing $u_{0n}^{(1)}, u_{0s}^{(1)}, \Phi_2, \partial\Phi_2/\partial n, A_k, B_p$ and $u_\xi^{(4)}$ as the independent variations of the boundary values of the functions, we obtain, as in [5], the relations defining the boundary conditions for the functions Φ_i, A_k and B_p and the integral condition for obtaining the induced potential difference φ which characterizes, to a certain extent, the interaction of the elastic and electric fields

$$a_{66}\Big[Q_n\Big(2\Phi_1 - \varepsilon^2\sum_{k=1}^{\infty}a_k^{(0)}A_k\Big)\Big]_{n=0} = N_0 \tag{3.2}$$

$$a_{66}\Big[Q_s\Big(2\Phi_1 - \varepsilon^2\sum_{k=1}^{\infty}a_k^{(0)}A_k\Big)\Big]_{n=0} = T_0$$

$$\Big\{a_{66}\frac{\partial}{\partial s}Q_s\Big[\frac{2}{3}\Phi_2 - \frac{8}{5}q_2\varepsilon^2\Delta_0\Phi_2 - \varepsilon^2\sum_{k=1}^{\infty}a_k^{(1)}A_k\Big] + \frac{1}{6}(a_{13}g_1 + b_{31}g_2 + a_{11})\frac{\partial}{\partial n}\Delta_0\Phi_2\Big\}_{n=0} + \varepsilon a_{44}\sum_{p=1}^{\infty}\delta_p^{-2}t_p^{(0)}(aR^{-1}S_p\beta_p + \varepsilon\beta_p'')' =$$

$$Z_0 + \frac{\partial}{\partial s}M_{ns}$$

$$\Big\{a_{66}Q_n\Big[\frac{2}{3}\Phi_2-\frac{8}{5}q_2\varepsilon^2\Delta_0\Phi_2-\varepsilon^2\sum_{k=1}^{\infty}a_k{}^{(1)}A_k\Big]-$$
$$\frac{1}{3}(a_{13}g_1+b_{31}g_2+a_{11})\,\Delta_0\Phi_2\Big\}_{n=0}+$$
$$\varepsilon a_{44}\sum_{p=1}^{\infty}\delta_p{}^{-2}t_p{}^{(0)}(S_p\beta_p'-\varepsilon aR^{-1}\beta_p')=M_{nn}$$

$$\Big\{2a_{66}\Big(S_m{}^*Q_n-\varepsilon\frac{\partial}{\partial s}Q_s\Big)[a_m^{(0)}\Phi_1+a_m^{(1)}\Phi_2+\varepsilon^2\Delta_0(q_0a_m^{(2)}\Phi_1+ \tag{3.3}$$
$$q_2a_m^{(3)}\Phi_2)]-a_m^{(4)}S_m^*\Delta_0\Phi_2+\varepsilon\omega_m^{(1)}\frac{\partial}{\partial n}\Delta_0\Phi_2-\varepsilon\gamma_m{}^2\theta_m^{(1)}\frac{\partial}{\partial n}\Delta_0\Phi_2\Big\}_{n=0}+$$
$$\sum_{k=1}^{\infty}(\gamma_k{}^2a_{mk}^{(1)}S_m{}^*+\gamma_k{}^2\omega_{mk}^{(1)}S_k-\gamma_m{}^2\theta_{mk}^{(1)}S_k)\,\alpha_k-$$
$$2a_{66}\varepsilon\sum_{k=1}^{\infty}a_{mk}^{(2)}S_m{}^*(aR^{-1}S_k\alpha_k+\varepsilon\alpha_k'')-\varepsilon^2 2a_{66}\sum_{k=1}^{\infty}a_{mk}^{(2)}(S_k\alpha_k'-$$
$$\varepsilon aR^{-1}\alpha_k')'+2a_{66}\varepsilon\sum_{p=1}^{\infty}a_{mp}^{(3)}S_m{}^*(S_p\beta_p'-\varepsilon aR^{-1}\beta_p')-$$
$$\varepsilon a_{66}\sum_{p=1}^{\infty}[-\delta_p{}^2a_{mp}^{(3)}\beta_p'+2\varepsilon a_{mp}^{(3)}(aR^{-1}S_p\beta_p+\varepsilon\beta_p'')']-$$
$$\varepsilon a_{44}\sum_{p=1}^{\infty}\omega_{mp}^{(2)}\beta_p'+\varepsilon b_{15}\gamma_m{}^2\sum_{p=1}^{\infty}\theta_{mp}^{(2)}\beta_p'=S_m{}^*N_m-\varepsilon T_m'-Z_m-$$
$$\gamma_m{}^2\lambda_m$$

$$\Big\{2a_{66}\Big(S_r{}^*Q_s-\varepsilon\frac{\partial}{\partial s}Q_n\Big)\Big[\frac{a_{44}}{a_{66}}\delta_r{}^{-2}t_r{}^{(0)}\Phi_2+\varepsilon^2\Delta_0(q_0t_r{}^{(1)}\Phi_1+ \tag{3.4}$$
$$q_2t_r{}^{(2)}\Phi_2)\Big]-\varepsilon t_r{}^{(3)}\frac{\partial}{\partial s}\Delta_0\Phi_2\Big\}_{n=0}+\varepsilon\sum_{k=1}^{\infty}[\gamma_k{}^2t_{rk}^{(0)}\alpha_k'-$$
$$\varepsilon 2a_{66}a_{kr}^{(3)}(aR^{-1}S_k\alpha_k+\varepsilon\alpha_k'')']+\varepsilon 2a_{66}\sum_{k=1}^{\infty}a_{kr}^{(3)}S_r{}^*(\varepsilon aR^{-1}\alpha_k'-$$
$$S_k\alpha_k')+S_r{}^*[-a_{66}\delta_r{}^2\beta_r+\varepsilon 2a_{66}(aR^{-1}S_r\beta_r+\varepsilon\beta_r'')]-$$
$$\varepsilon 2a_{66}(\varepsilon aR^{-1}\beta_r'-S_r\beta_r')'=\varepsilon N_r{}^\circ+S_r{}^*T_r{}^\circ$$

$$\varphi=-\frac{2}{\varkappa_5\Pi}\iint_{S_+}\Big[\varkappa_6\Delta_0\Phi_1+\sum_{k=1}^{\infty}\gamma_k{}^2(q_7f_k''+q_{12}\theta_k')A_k\Big]d\Sigma \tag{3.5}$$

where

$$a_k^{(i)}=\int_{-1}^{1}a_kP_i\,d\xi,\ \ i=0,1,2;\ \ a_k{}^{(3)}=\int_{-1}^{1}a_kF\,d\xi,\ \ a_m{}^{(4)}=\int_{-1}^{1}\varkappa_2a_m\,d\xi$$

$$a_{mk}^{(1)}=\int_{-1}^{1}f_k''a_m\,d\xi,\ \ a_{mk}^{(2)}=\int_{-1}^{1}a_ka_m\,d\xi,\ \ a_{mp}^{(3)}=\int_{-1}^{1}t_pa_m\,d\xi$$

$$\omega_m{}^{(1)} = \int_{-1}^{1} \varkappa_3\omega_m\, d\xi,\ \ \omega_{mk}^{(1)} = \int_{-1}^{1} f_k{}'\omega_m\, d\xi,\ \ \omega_{mp}^{(2)} = \int_{-1}^{1} t_p{}'\omega_m\, d\xi$$

$$\theta_m^{(1)} = \int_{-1}^{1} \varkappa_4\theta_m\, d\xi,\ \ \theta_{mk}^{(1)} = \int_{-1}^{1} d_k\theta_m\, d\xi,\ \ \theta_{mp}^{(2)} = \int_{-1}^{1} t_p{}'\theta_m\, d\xi,\ \ t_p{}^{(0)} = \int_{-1}^{1} t_p{}'\, d\xi$$

$$t_r{}^{(1)} = \int_{-1}^{1} t_r P_2\, d\xi,\ \ t_r{}^{(2)} = \int_{-1}^{1} t_r F\, d\xi,\ \ t_r{}^{(3)} = \int_{-1}^{1} \varkappa_2 t_r\, d\xi,\ \ t_{rk}^{(0)} = \int_{-1}^{1} f_k{}'' t_r\, d\xi$$

$$2\varepsilon N_0 = \int_{-1}^{1} N\, d\xi - \varepsilon\varkappa_1\varphi, \quad 2\varepsilon T_0 = \int_{-1}^{1} T\, d\xi, \quad 2\varepsilon^2 Z_0 = \int_{-1}^{1} Z\, d\xi$$

$$2\varepsilon M_{ns} = \int_{-1}^{1} T\xi\, d\xi, \quad 2\varepsilon M_{nn} = \int_{-1}^{1} N\xi\, d\xi, \quad \varepsilon N_m = \int_{-1}^{1} N a_m\, d\xi - \frac{\varkappa_1}{2} a_m^{(0)}\varphi$$

$$\varepsilon T_m = \int_{-1}^{1} T a_m\, d\xi, \quad \varepsilon^2 Z_m = \int_{-1}^{1} Z\omega_m\, d\xi, \quad \varepsilon^2\lambda_m = \int_{-1}^{1} \lambda\theta_m\, d\xi$$

$$\varepsilon N_r{}^{\circ} = \int_{-1}^{1} N t_r\, d\xi, \quad \varepsilon T_r{}^{\circ} = \int_{-1}^{1} T t_r\, d\xi$$

$\alpha_k(s)$ and $\beta_p(s)$ are the boundary values of the functions A_k and B_p on ∂S, S_k is an operator introduced according to the rule [9] $S_k\alpha_k = \varepsilon\partial A_k / \partial n$ and $S_k{}^*$ is its conjugate, and Π is the area of S_+.

From (3.5) it follows that the induced potential difference φ is connected only with the biharmonic and potential parts of the deformation of the plate symmetrical relative to the middle surface. This fact becomes obvious in the case of the inverse piezo-effect. An application of electric potential difference to the plate end electrodes cannot produce bending, nor torsional deformation.

If we use (3.3) and (3.4) to eliminate from (3.2) the functions α_k and β_p we obtain at once the boundary conditions for the functions Φ_i, which determine the internal electroelastic state of the plate.

4. Taking ε as a small parameter, we seek the solution of Eqs. (3.2—(3.5) in the form of the following series [5]:

$$\Phi_i = \Phi_{i0} + \varepsilon\Phi_{i1} + \ldots, \qquad \alpha_k(s) = \alpha_{k0} + \varepsilon\alpha_{k1} + \ldots$$
$$\beta_p(s) = \beta_{p0} + \varepsilon\beta_{p1} + \ldots, \qquad \varphi = \varphi_0 + \varepsilon\varphi_1 + \ldots$$

We can assume here that the external physical factors acting on Γ can be represented in the form

$$N(s, \xi) = \varepsilon\,(N^{(0)} + \varepsilon N^{(1)} + \ldots), \quad T(s, \xi) = \varepsilon\,(T^{(0)} + \varepsilon T^{(1)} + \ldots)$$

$$Z(s,\ \xi) = \varepsilon^2 (Z^{(0)} + \varepsilon Z^{(1)} + \ldots),\quad \lambda(s,\ \xi) = \varepsilon^2 (\lambda^{(0)} + \varepsilon\lambda^{(1)} + \ldots)$$

and are sufficiently smooth, slowly varying functions of s.

Using the asymptotic expansions of the operators S_k and S_k^* [9] we obtain, from (3.2)—(3.5) the boundary conditions for the functions Φ_i, A_k, B_p and the constant φ, in every approximation in ε. In zero approximation we find

$$2a_{66}\ [Q_n\Phi_{10}]_{n=0} = N_0^{(0)},\quad 2a_{66}\ [Q_s\Phi_{10}]_{n=0} = T_0^{(0)} \tag{4.1}$$

$$\left[\frac{2}{3}a_{66}\frac{\partial}{\partial s}Q_s\Phi_{20} + \frac{1}{6}(a_{13}g_1 + b_{31}g_2 + a_{11})\frac{\partial}{\partial n}\Delta_0\Phi_{20}\right]_{n=0} = Z_0^{(0)} + \frac{\partial}{\partial s}M_{ns}^{(0)} \tag{4.2}$$

$$\left[\frac{2}{3}a_{66}Q_n\Phi_{20} - \frac{1}{3}(a_{13}g_1 + b_{31}g_2 + a_{11})\Delta_0\Phi_{20}\right]_{n=0} = M_{nn}^{(0)}$$

$$\sum_{k=1}^{\infty}(\gamma_k^2\gamma_m a_{mk}^{(1)} + \gamma_k^3\omega_{mk}^{(1)} - \gamma_m^2\gamma_k\theta_{mk}^{(1)})\alpha_{k0} = \gamma_m N_m^{(0)} - Z_m^{(0)} - \tag{4.3}$$
$$\gamma_m^2\lambda_m^{(0)} - 2a_{66}\gamma_m [Q_n(a_m^{(0)}\Phi_{10} + a_m^{(1)}\Phi_{20})]_{n=0} + \gamma_m a_m^{(4)}[\Delta_0\Phi_{20}]_{n=0}$$

$$\beta_{r0} = -(a_{66}\delta_r^2)^{-1}[T_r^{\circ(0)} - 2a_{44}\delta_r^{-2}t_r^{(0)}Q_s\Phi_{20}]_{n=0}$$

$$\varphi_0 = -\frac{2\varkappa_6}{\varkappa_5\Pi}\iint_{S_1}\Delta_0\Phi_{10}d\varepsilon$$

Thus the internal electroelastic state of the plate is determined, with the accuracy of the order of ε, from the boundary value problems (4.1) and (4.2) which are equivalent to the plane problem of the theory of elasticity and of the problem of bending.

Expressing the biharmonic function Φ_1 in terms of the analytic functions $\varphi(z)$ and $\psi(z)$

$$4a_{66}\Phi_1 = \bar{z}\varphi + z\bar{\varphi} + \chi(z) + \overline{\chi(z)},\ d\chi/dz = \psi(z) \qquad (z = \xi_1 + i\xi_2)$$

we can write the condition (4.1) in the classical form [10]

$$d/ds\ (\varphi_0 + z\,\bar{\varphi}_0' + \bar{\psi}_0) = i\,(X_{n0}^{(0)} + iY_{n0}^{(0)})$$

$$X_{n0}^{(0)} = \frac{1}{2}\int_{-1}^{1}(N_0^{(0)}l - T_0^{(0)}m)\,d\xi,\quad Y_{n0}^{(0)} = \frac{1}{2}\int_{-1}^{1}(N_0^{(0)}m + T_0^{(0)}l)\,d\xi$$

$$l = \cos(n,\ \xi_1),\quad m = \cos(n,\ \xi_2)$$

It is clear that the matrix of the infinite system (4.3) is independent of the load and the plate geometry, and remains the same in all approximations in ε.

As in the theory of elastic plates [3, 5, 9], the potential and rotational solutions in terms of the stresses σ_n, σ_s, σ_{ns}, and in the present case also in terms of D_ξ, are of the same order in ε as the biharmonic solution. Moreover, the boundary layer solutions determine the behavior of σ_ξ, $\sigma_{\xi s}$, $\sigma_{\xi n}$, D_n and D_s on Γ and the latter are found to be of the same order in ε as σ_n, σ_s, σ_{ns}, and D_ξ.

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